OPTIMAL STRUCTURAL DESIGN OF DYNAMICALLY EXCITED SYSTEMS USING THE FINITE-ELEMENT-METHOD*

D.W. Mathias and H. Röhrle
Dornier GmbH
Friedrichshafen Germany

Abstract

This report explains the application of the finite-element method to the weight-optimal design of harmonically excited structures with viscous damping. The following constraints can be taken into account: Maximum stresses in the elements, maximum accelerations and maximum deformations at the degrees of freedom. The optimization task is solved according to the fully-stressed design and the gradient method. Both procedures can be treated one after the other in one calculation process. That way the advantages of the rapid initial convergence of the fully-stressed design are combined with the mathematically exact optimization according to the gradient method. The described proceeding leads to good convergence.

One important advantage of the gradient method concerning structural dynamics, in particular, is founded on the fact that further constraints can be introduced into the program in a simple way. Experiences have already been made with the constraint "minimum natural frequency of a system" and, in the field of wing design, with the constraint "minimum flutter speed".

The dynamic optimization program DYNOPT was applied to the AEROS satellite. Compared with a version manufactured for the structural test weight savings of about 20 % could be achieved. The considered mathematical model consists of 18 degrees of freedom and 21 elements. The number of the used constraints amounted to 57. The final result was achieved after 30 iterations.

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I. Introduction

All optimization methods mentioned in the literature are iterative procedures; this means that the optimum structural design which meets the requirements can only be achieved by a sequence of identical calculation operations. Good convergence behaviour and economic efficiency are the prerequisites for the applicability of a procedure to a structure optimization task. The procedures which by now have largely been used for static problems were examined from these points of view. It is the objective of this work to show the application and enlargement of these procedures to dynamic structural problems.

Basically, there are two types of optimization methods:

a. Procedures based on optimality criteria

b. Mathematical procedures.

With the methods mentioned first, criteria are elaborated which, according to the engineer's experience may result in a favourable structual design which, however, does not necessarily represent the optimum. The optimality criterion is supposed to supply a structural design which is also sufficient for the purposes of the design engineer. Among the number of procedures working with optimality criteria the fully-stressed design is suitable for the optimization of dynamically excited systems.

As a mathematical procedure, the gradient method is mentioned which linearizes the optimization problem step by step. On this basis the optimization task can be represented mathematically exact. The weight function which is to be minimized, and the constraints are processed together in each iteration step by means of the simplex algorithm. The obtained optimum is exact within the scope of the desired accuracy.

II. Assumptions and Basic Equations

The task consists in designing harmonically excited structures with viscous damping that are optimal with respect to weight. The constraints require that neither the maximum accelerations and maximum deformations at the degrees of freedom nor the maximum stresses in the elements may be exceeded. The design variables are the element stiffness values and the damping values. The geometry of the system remains unchanged in the course of optimization, i.e. the elements do not disappear. The damping values may, however, assume any positive value or even become zero. Because systems with arbitrary viscous damping (dashpots) have to be calculated, the frequency response method is suitable for the determination of the dynamic response.

The equations of motion may be expressed as follows in the form of matrices:

$$\underline{M}\cdot\underline{\ddot{q}\ (t)} + \underline{C}\cdot\underline{\dot{q}\ (t)} + \underline{K}\cdot\underline{q\ (t)} = \underline{R\ \ (t)}, \qquad (1)$$

with $\underline{M}$ as mass matrix, $\underline{C}$ as damping matrix, and $\underline{K}$ as stiffness matrix. The vector $\underline{q\ (t)}$ contains the time-dependent displacements of the system. The vectors $\underline{\dot{q}}$ and $\underline{\ddot{q}\ (t)}$ are the derivations with respect to time of the deformation vector $\underline{q\ (t)}$.

The structure is excited by a harmonic force.

$$\underline{R\ (t)} = \underline{R}_o\cdot e^{i\Omega t} \qquad (2)$$

In the steady state the system response is also harmonic and given by the frequency of excitation.

$$\underline{q\ (t)} = \underline{r}_o e^{i\Omega t} \qquad (3)$$

with $\underline{r}_o$ as the complex response amplitude vector.

This vector is calculated according to the frequency response method:

$$\underline{r}_o = \underline{Z}^{-1}\cdot \underline{R}_o, \qquad (4)$$

with the impedance matrix:

$$\underline{Z} = \underline{K} - \Omega^2\ \underline{M} + i\Omega\underline{C} \qquad (5)$$

The maximum values of the accelerations and deformations which are important for the problem can be determined by means of the statement of motion (3).

The maximum stresses are related to the respective element coordinate system and result from the equation:

$$\underline{\sigma} = \underline{D}\cdot\underline{T}\cdot\underline{r}_o, \qquad (6)$$

with $\underline{D}$ as difference matrix containing the values $\frac{E_j}{l_j}$ provided with signs (E = modulus of elasticity, l = length of the elements) for the j-th tension-compression bar, for example. The matrix $\underline{T}$ establishes the relation between the basis and element coordinates.

The system matrices $\underline{M}$, $\underline{C}$ and $\underline{K}$ in (1) result from the superposition of the respective element matrices. They describe the inertia, damping, and stiffness distribution in the structure. Since the stiffness and damping values represent the variable design quantities, the system matrices have to be established anew after each improvement of the structure.

The system mass changes also from one iteration step to the other together with the design variables. In the case of a diagonal mass matrix $\underline{M}$ for a system composed of bars the masses concentrated in the knots have the quantity:

$$m_i = m_{oi} + \frac{1}{2}\sum_j (\rho_j\cdot A_j\cdot \nu k_j\cdot l_j + \nu c_j\ \underline{C}_j), \qquad (7)$$

with ρ = material density
A = cross-sectional area

The index j characterizes all elements joining in the i-th knot. The device masses which are invariable in the course of optimization are referred to as m_{oi}. ν_k and ν_c designate mass transmission factors which indicate the ratio of the design element mass to the theoretical element mass.

III. Preliminary Layout According to the Fully-Stressed Design

The procedure of the fully-stressed design is based on the assumption that the structural weight is minimized when the permissible stresses in the individual elements are optimally used. A mathematical formulation for the weight minimum can not be introduced in the calculation process. The method of the fully-stressed design may be applied to dynamic problems. However, only constraints can be considered in the element stresses, and, as design variables only the cross-section values are available. The acceleration and deformation constraints with simultaneous damping variation can only be observed in separate calculation processes. In general, the convergence is good, it grows worse, however, with the structures becoming more complex. For these reasons the fully-stressed design is used only for a preliminary optimization of dynamically stressed structures. The element cross-section values A are improved corresponding to the ratio of the existing stresses to the permissible stresses according to:

$$A_j^{(\nu+1)} = A_j^{(\nu)} \cdot \left(\frac{\sigma_{j\ vorh}^{(\nu)}}{\sigma_{j\ zul}}\right)^{\beta} \quad \text{with } 1{,}0 \leq \beta \leq 1{,}2 \tag{8}$$

for the j-th element in the ν-th iteration step. The rearrangement of the forces in statically indeterminate structures is taken into account by the exponent β which influences the convergence speed. For values $\beta > 1.2$ the cross sections may oscillate violently with the iterations; or divergence may occur. The same may happen in applying the fully-stressed design when the structural layout is not far from its weight minimum.

IV. Improvement of the Structural Design According to the Gradient Method

The dynamic response of an excited structure depends on the size of its stiffness and damping values k and c. for the i-th degrees of freedom and the j-th element this results in:

$$\ddot{q}_{i_{max}} = \ddot{q}_{i_{max}}(k_m, c_m)$$

$$q_{i_{max}} = q_{i_{max}}(k_m, c_m) \qquad \text{m applies to all elements} \tag{9}$$

$$\sigma_{j_{max}} = \sigma_{j_{max}}(k_m, c_m)$$

The structural weight is a function of the design variables:

$$W = W(k_m, c_m) \tag{10}$$

The changes of the dynamic response in dependence of the design variables improvements dx can be expressed in an abbreviated form as follows:

$$d\ddot{q}_i = \underline{(\nabla\ddot{q})^t}\ \underline{dx} \tag{11}$$

The requirement that accelerations, deformations, and stresses must not exceed fixed maximum values leads to the constraints system:

$$\left.\begin{aligned} \underline{\ddot{q}}^{(\nu)} + \underline{((\nabla\ddot{q})^{(\nu+1)})^t} \cdot \underline{\Delta x}^{(\nu+1)} &\leq \underline{\ddot{q}_{zul}} \\ \underline{q}^{(\nu)} + \underline{((\nabla q)^{(\nu+1)})^t} \cdot \underline{\Delta x}^{(\nu+1)} &\leq \underline{q_{zul}} \\ \underline{\sigma}^{(\nu)} + \underline{((\nabla \sigma)^{(\nu+1)})^t} \cdot \underline{\Delta x}^{(\nu+1)} &\leq \underline{\sigma_{zul}} \end{aligned}\right\} \tag{12}$$

The following equation applies to the weight function accordingly:

$$W^{(\nu+1)} = W^{(\nu)} + \underline{(\nabla W)^t} \cdot \underline{\Delta x}^{(\nu+1)} \stackrel{!}{=} \text{Min.} \tag{13}$$

The index $(\nu+1)$ stands for the $(\nu+1)$-th iteration step.

The deformation gradient is obtained by applying the ∇-operator to the relation (4).

$$\underline{\nabla r_o} = -\,(\underline{K} - \Omega^2\underline{M} + i\Omega\underline{C})^{-1} \cdot (\nabla\underline{K} - \Omega^2\nabla\underline{M} + i\Omega\nabla\underline{C}) \cdot \underline{r_o} \tag{14}$$

According to the equation of deformation (3) the acceleration gradient is determined by:

$$\underline{\nabla\ddot{q}} = -\,\Omega^2\,\underline{\nabla r_o} \tag{15}$$

and, according to (6) the stress gradient results in:

$$\underline{\nabla\sigma} = \underline{D} \cdot \underline{T} \cdot \underline{\nabla r_o} \quad (16)$$

The number of lines of the gradients is equal to the number of the respective constraints, the number of columns is given by the number of the design variables.

The size of the gradient system can be reduced if, during the calculation process, a difference is made between the active and passive constraints. In this case, the constraints, having only little or even no influence on the system layout, are eliminated at each iteration step. The active constraints are selected from the overall constraints system by the transformation:

$$\underset{(nxm)}{\underline{G}^t} = \underset{(nxl)}{\underline{S}^t} \underset{(lxm)}{(\underline{\nabla\ddot{q}}, \underline{\nabla q}, \underline{\nabla\sigma})^t} \quad (17)$$

In this equation, $\underline{S}^t$ is a Boole's matrix whose number of columns l represents the total number of all existing constraints, and whose number of lines n corresponds to the number of the active constraints. The matrix $\underline{G}^t$ is the reduced gradient whose number of columns m indicates the number of the design variables.

In its final form the restriction system can be described as follows:

$$\underline{G}^t \cdot \underline{\Delta x} \lesseqgtr \underline{r} \quad \text{with} \quad \underline{\Delta x}_{ll} \leq \underline{\Delta x} \leq \underline{\Delta x}_{ul} \quad (17)$$

$\underline{\Delta x}$ is the vector of the unknown changes of the design variables. The indices ll and ul stand for "lower limit" and "upper limit". The vector $\underline{r}$ contains the greatest possible increases of the limited values in the respective iteration step.
In the next step the weight function (13) is minimized while the boundary conditions defined in (17) are observed. To this end, the Simplex algorithm is used.

The applicability of the simplex procedure depends on the following conditions:

a) More variables Δx than equations must be available.

b) The variables Δx must be greater than or equal to zero.

Condition a) is not met only if the system is subjected to too many constraints which consequently leads to the incompatibility of two or several constraints. A solution to this problem is the examination of incompatible constraints within the restriction system in order to eliminate them.

Condition b) is always met by means of a transformation.

$$\underline{0} \leq \underline{\overline{\Delta x}} \leq \underline{\Delta x}_{ul} - \underline{\Delta x}_{ll} \quad \text{with} \quad \underline{\overline{\Delta x}} = \underline{\Delta x} - \underline{\Delta x}_{ll} \quad (18)$$

The true values of the changes of the design variables result from the re-transformation of $\underline{\overline{\Delta x}}$.

In the case of a geometrical interpretation of the Simplex algorithm the restrictions represent the planes limiting a space of permissible solutions. The origin of coordinates must be included therein, according to the requirement that the solution point components assume values that are greater than or equal to zero.

The object function - in the present case the weight function - is a plane which is shifted against the space of the permissible solutions while the direction of its normal vectors remains the same. The direction of the shift results from the requirement of a minimum value of the object function. The solution is reached when the object function will touch the space of the permissible solutions in one of its corners.

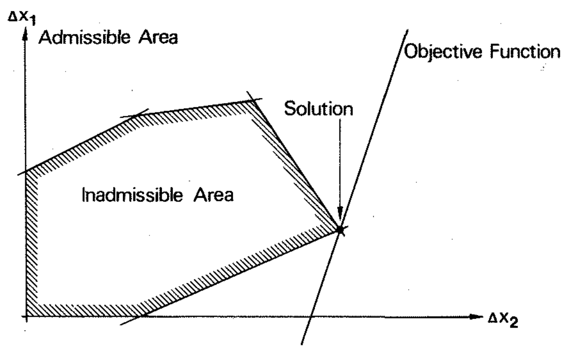

FIGURE 1 - Geometric Description of the Simplex Algorithm

After the run of the Simplex algorithm the design variables and the structural weight will be corected:

$$F_j^{(\nu+1)} = F_j^{(\nu)} + \Delta F_j^{(\nu+1)}$$

$$C_j^{(\nu+1)} = C_j^{(\nu)} + \Delta C_j^{(\nu+1)} \qquad (19)$$

$$W^{(\nu+1)} = W^{(\nu)} + \Delta W^{(\nu+1)}$$

The iteration procedure is concluded once the optimum has been reached. Here, it is necessary that the weight changes by no more than one given percentage in the case of a monotone approach to its minimum.

V. Sequence of Operations within the Optimization Package

The gradient method and the "fully-stressed design" are combined with other computer programs in a mathematical optimization procedure. Fig. 2 shows a rough flow diagram. For each new system configuration one run through the great iteration loop is performed. This iteration loop includes the drawing up of the improved system matrices, a dynamic analysis, and an evaluation of the eigenvalues in which the actual values are calculated. The loop also includes one of the described optimization procedures. The calculation is concluded with the question for convergence. Thereupon either another iteration step is made or the calculation is concluded, provided the accuracy is adequate.

The calculation run,which begins with a set of initial values for the stiffnesses and the damping coefficients,is fully automatic. Any interventions of the user are not necessary.

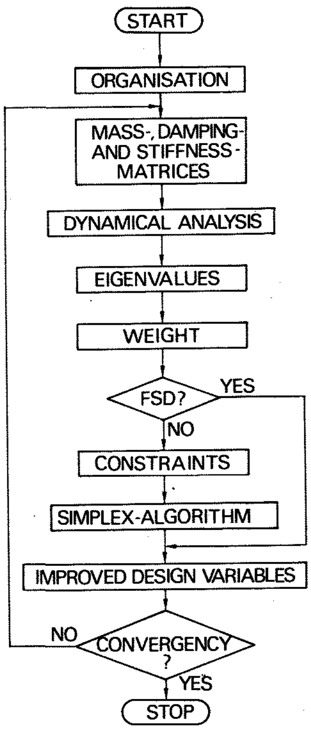

FIGURE 2 - Diagram of DYNOPT Calculation

VI. Application of the DYNOPT to the Satellite AEROS

The program DYNOPT has been applied to two calculation models of the satellite AEROS. These are the longitudinal vibration models with 8 and 18 degrees of freedom respectively. Due to symmetry the calculation can be carried out with one half of the system, i.e. in the following diagrams only one half of the structural weight is given. Apart from the number of the degrees of freedom the two systems also differ in their structure. The model of 8 degrees of freedom is statically determinate. All spring-mass-chains are open. The model with 18 degrees of freedom has 3 self-contained spring-mass-chains and is statically indeterminate. Fig. 3 shows the aeronomy satellite AEROS and in Fig. 4 a graph of the mathematical model with 8 degrees of freedom is given.

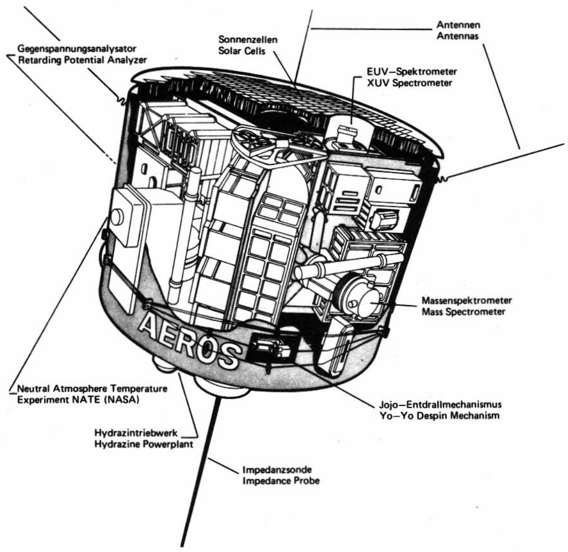

FIGURE 3 - Aeronomiesatellite AEROS

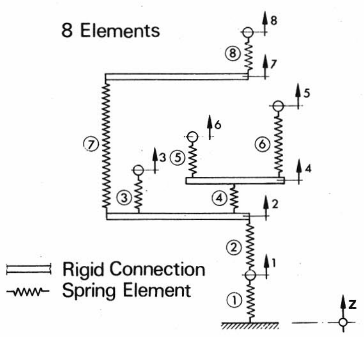

FIGURE 4 - 8 DOF model of the satellite AEROS

The damped system is pre-optimized according to the "fully-stressed design" and its weight is optimized by means of the gradient method. The "fully-stressed design" allows only the stiffnesses as design variables. Fig. 5 shows how fast the stiffnesses change with the iteration steps.

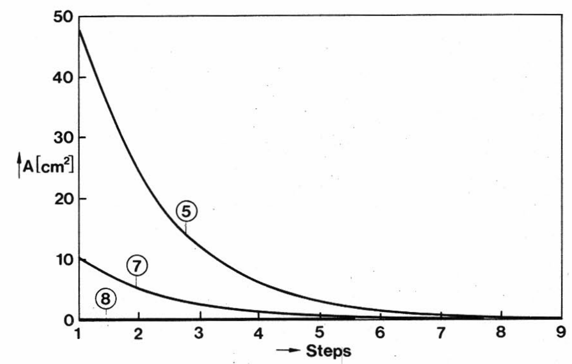

FIGURE 5 - Cross-sections of some elements

In Fig. 6 the changes of the normal stress in element ① are shown. This element is the only one in which the permissible stress of 10^4 N/cm^2 is reached.

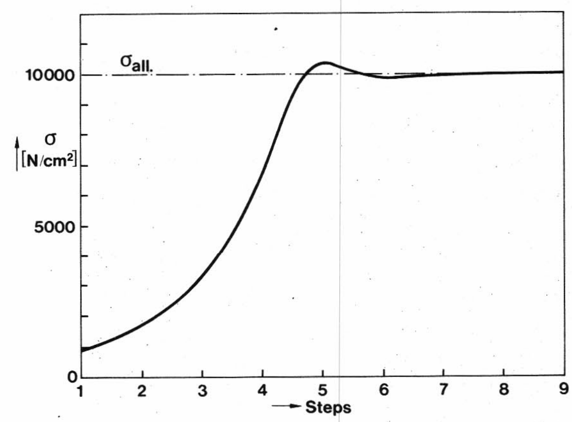

FIGURE 6 - Stress in element ①

Fig. 7 shows the weight changes above the iteration steps. The structural weight approaches in monotone convergence its optimum which is reached already after 9 iteration steps. The weight savings amount to about 20 %.

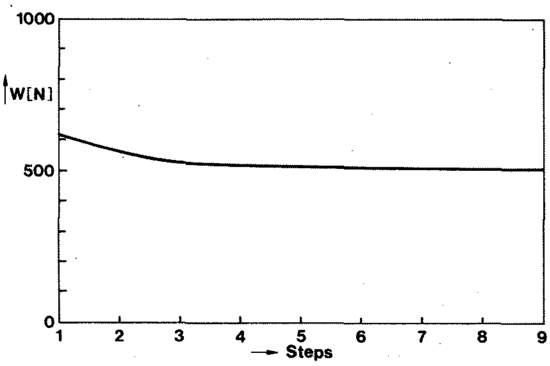

FIGURE 7 - Weight of one half of the satellite AEROS

The following diagrams include the results of the optimization calculation with the mathematical model with 18 degrees of freedom. Only the gradient method was applied. Here, 36 stiffness values and damping coefficients are variable and 51 stress, deformation, and acceleration constraints are given. The system is excited in its respective fundamental frequency by means of harmonic forces. In order to reduce the number of calculation processes an activation limit of 2o % has been introduced. Thus,only those restrictions are used for which the described value is 20 % greater than permissible. Compared to the calculation with the model with 8 degrees of freedom the permissible minimum cross-sections have been increased so,that the calculation produces cross-sections that are compatible with the design. Furthermore, reduced permissible stresses are calculated with in order to design the elements according to the criteria of stability. Both measures lead to a somewhat higher structural weight.

Fig. 8 shows the mathematical model with 18 degrees of freedom which is used for the satellite AEROS.

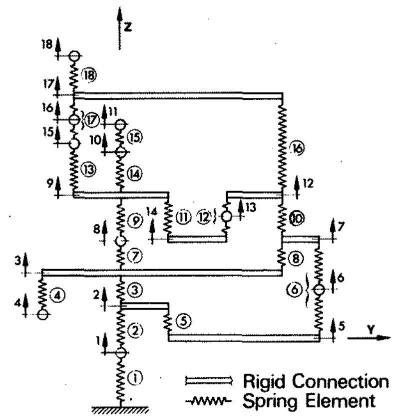

FIGURE 8 - 18 DOF model of the satellite AEROS

With each element which is identified as spring a damper is also connected in parallel.

Since the accelerations and deformations are far below the permissible limits only some of the stresses are shown as example in Fig. 9.

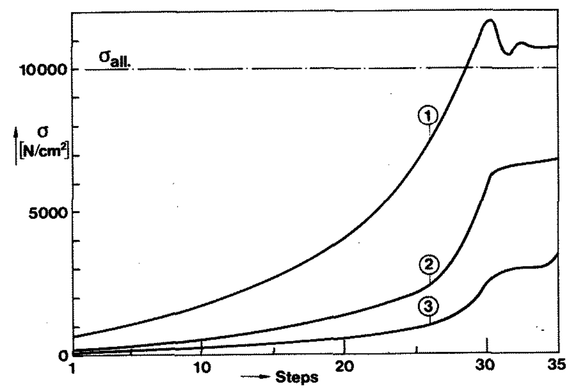

FIGURE 9 - Stress in elements ① to ③

The stress of element ① is slightly higher than the permissible maximum value of 10^4 N/cm^2 and oscillates only within a very limited range. This phenomenon sometimes occurs if several constraints are exceeded simultaneously, especially in statically indeterminate systems. This is due to the fact that in the closed circuit systems of springs and dampers the weight is constantly shifted.
In Fig. 10 the curve of the fundamental frequency is plotted above the iteration steps.

VII. Conclusion

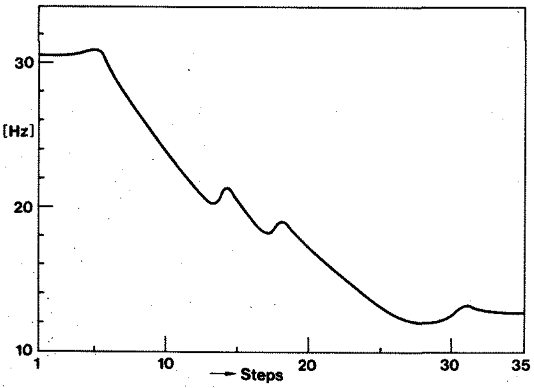

FIGURE 10 - Lowest fundamental frequency

As already seen in the case of the stresses the system starts to stabilize with the 30th iteration step. The subsequently occurring changes of the system responses as well as of the design variables are insignificant.

Fig. 11 shows the curve of the weight changes which is plotted above the number of iterations.

The values decrease monotonously and approach thus the minimum.

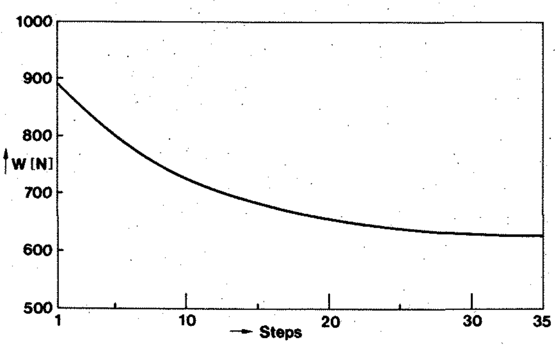

FIGURE 11 - Weight of one half of the satellite AEROS, 18 DOF

The convergence is good. Numerical difficulties did not occur. The weight savings as against the first realistic structural design amount to about 25 %.

A description was given of a general optimization procedure for the weight-optimal design of dynamically excited structures with which large-scale optimization calculations are carried out for satellites.

The basis of this method, i.e. the "Finite-Element-Method" and the "Gradient-Method", allows the extension to any number of element types. In addition to constraints concerning accelerations, deformations, and stresses, other constraints can be included. Thus, weight optimization calculations under constraints concerning minimum natural frequencies were carried out in individual examinations. In the field of wing design for aircraft wing boxes were designed with an optimum weight in consideration of statical and aeroelastic constraints. The integration of the mentioned constraints into the entire program is easily possible.